the identification of T_4^4 with the mass density, and admitting in our relativistic treatment that the total mass of the universe might depend on the mean radius of curvature of its space.[31] We may interpret this dependence either as the creation of matter possessing invariable gravitational properties, or as a variation of the gravitational properties of matter (in the sense of mutual action of matter proposed by Einstein)[19] the total quantity of which remains constant in the universe. The latter variation is caused by the variation of the mass of matter, in contra-distinction to the hypothesis of Dirac[32] who assumed a dependence of the gravitational "constant" on the radius of the universe. Whether such variations do occur in our universe or not, only experience can decide. The recent observations of Ambarzumian,[33] who found that the central regions of certain galaxies are the sources of an intensive emanation of matter, indicate that such a possibility cannot be *a priori* excluded.

[31] Consequences of the assumption that the mean-mass density varies as the function $\rho=\rho_1(G_1/G)^{3+n}$, ρ_1 being the density at the radius G_1, and n a real constant, are investigated in J. Pachner, Acta Phys. Polon. **23**, 133 (1963).

[32] P. A. M. Dirac, Proc. Roy. Soc. (London) **A165**, 199 (1938).

[33] V. A. Ambarzumian, *Voprosy kosmogonii*, tom VIII (Moscow, 1962), pp. 21–23.

PHYSICAL REVIEW VOLUME 132, NUMBER 4 15 NOVEMBER 1963

Lorentz-Covariant Position Operators for Spinning Particles*

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(Received 2 July 1963)

An examination is made of the consequences for the quantum mechanics of spinning particles of equations characteristic of Lorentz-covariant position variables. These equations are commutator analogs of the Poisson bracket equations that express the familiar transformation properties of space-time events in classical mechanics. For a particle of zero spin it is found that the usual canonical coordinate is the unique solution of these equations. For a particle with positive spin there is no position operator which satisfies these equations and has commuting components. For a particle and antiparticle there is a unique solution with commuting components which is valid for all values of the spin and reduces for zero spin to the canonical coordinate. For spin $\frac{1}{2}$ this is the Foldy-Wouthuysen transform of the position operator of the Dirac equation. A generalization of the inverse Foldy-Wouthuysen transformation, valid for any value of the spin, appears as a unique unitary transformation which takes this generalized Dirac position to the canonical coordinate. The application of this transformation to the canonical form of the Hamiltonian gives a generalization of the Dirac equation Hamiltonian. This is developed and compared with the literature for spin 1. It gives a nonlocal equation as the spin 1 analog of the Dirac equation.

I. INTRODUCTION

THIS paper is an attempt to answer some questions suggested by a recent study of special relativistic invariance in Hamiltonian particle dynamics.[1,2] This study has emphasized two distinct aspects of relativistic invariance. The first of these is the symmetry of the theory under the inhomogeneous Lorentz group, reflecting the principle of special relativity that the laws of physics should be invariant under transformations of reference frames. This symmetry is guaranteed by postulating the existence of ten infinitesimal generators H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$, for time translations, space translations, space rotations, and pure Lorentz transformations, respectively, satisfying the bracket equations

$$\begin{aligned}
&[P_j,P_k]=0\,, \quad [P_j,H]=0\,, \quad [J_k,H]=0\,,\\
&[J_i,J_j]=\epsilon_{ijk}J_k\,, \qquad [J_i,P_j]=\epsilon_{ijk}P_k\,,\\
&[J_i,K_j]=\epsilon_{ijk}K_k\,, \qquad [K_j,H]=P_j\,,\\
&[K_i,K_j]=-\epsilon_{ijk}J_k\,, \qquad [K_j,P_k]=\delta_{jk}H
\end{aligned} \tag{A}$$

which are characteristic of the inhomogeneous Lorentz group.[1,3] (We choose units in which $\hbar=c=1$. The summation convention is used for the indices $i, j, k=1,2,3$. In classical mechanics the brackets are Poisson brackets. In quantum mechanics they are commutators divided by i. This notation is maintained throughout the paper.)

The second aspect involves the explicit transformation properties of space-time events and gives the

* Supported in part by the U. S. Atomic Energy Commission.

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[1] D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, Rev. Mod. Phys. **35**, 350 (1963).

[2] D. G. Currie, University of Rochester Report NYO-10242 (to be published); and thesis, University of Rochester, 1962 (unpublished).

[3] P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).

equations

$$[x_j,P_k]=\delta_{jk},$$
$$[J_i,x_j]=\epsilon_{ijk}x_k,$$
$$[x_j,K_k]=\tfrac{1}{2}(x_k[x_j,H]+[x_j,H]x_k), \qquad \text{(B)}$$

to be satisfied by Cartesian particle position variables **x**. In classical mechanics Eqs. (B) are necessary and sufficient conditions for the time-dependent values of the position variables to transform in the familiar manner of space-time events under space translations, space rotations, and pure Lorentz transformations.[1] In particular, the last equation, which is the least familiar, is equivalent to the Lorentz transformation formula. The first two of Eqs. (B) have a similar meaning in quantum mechanics, but the status of the last equation in quantum mechanics is not so clear.[1] Equations (A) and (B) have been used to prove theorems that there can be no interaction in a classical mechanical system of two or three particles.[1,4] These theorems have dealt only with particles of zero spin.

In the present paper we seek to determine the role of Eqs. (B) in quantum mechanics, especially the role of the last of Eqs. (B) in the quantum mechanics of particles with spin. To begin, we confine our attention to a single free particle of positive mass m and spin s; we take H, **P**, **J**, **K** to be the Hermitian operators which are generators of the irreducible unitary representation of the inhomogeneous Lorentz group which is characterized by mass m and spin s. We work with canonical forms for these generators in terms of canonical coordinate, momentum, and spin operators **q**, **p**, and **S** and look for operators **x** satisfying Eqs. (B).

In Sec. II we find that for zero spin Eqs. (B) have a unique solution for **x**. With the canonical forms for H, **P**, **J**, **K**, this is just the canonical coordinate **x**=**q**. It is essentially the same as the position operator found by Newton and Wigner[5] and also agrees with various definitions of Pryce.[6] For zero spin it is no problem to find a satisfactory position operator satisfying our conditions (B) for relativistic covariance. The solution is simple and has been found by a variety of approaches.

In Sec. III we find that for positive spin Eqs. (B) have a one-parameter family of solutions for **x**. All of these solutions fail to satisfy the equations

$$[x_j,x_k]=0. \qquad \text{(C)}$$

Since it is desirable for many purposes to have a position operator whose different components commute, we conclude that it is not possible to have a completely satisfactory position operator satisfying the conditions (B) for relativistic covariance for a single particle of positive mass and positive spin. This is again in accord with findings of Newton and Wigner[5] and of Pryce.[6]

[4] J. T. Cannon and T. F. Jordan, University of Rochester Report NYO-10263 (to be published).

[5] T. D. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 400 (1949).

[6] M. H. L. Pryce, Proc. Roy. Soc. (London) **A195**, 62 (1948).

In Sec. IV we expand the scope of our investigation to include antiparticle as well as particle states. Equations (B) and (C) then do have a solution. It is the Foldy-Wouthuysen transform[7] of the operators **x** that appear as independent variables in the Dirac equation. This turns out to be a solution of Eqs. (B) and (C) for all values of the spin, not just for spin 1/2. It reduces to the canonical coordinates **x**=**q** for zero spin. Furthermore, it is the unique solution of Eqs. (B) and (C) having these properties. Equations (B) and (C) lead us in a rather unique manner to operators **x** which generalize the canonical position for zero spin and the position operators of the spin-1/2 Dirac equation to any value of the spin.

The generalized Dirac position operators lead us, in turn, to a generalization of the Foldy-Wouthuysen transformation[7] which is valid for any value of the spin. The generalized inverse Foldy-Wouthuysen transformation appears as the essentially unique unitary transformation which takes the generalized Dirac position operator to the canonical position **x**=**q** while leaving the canonical forms for **P** and **J** unchanged.

In Sec. V the generalized inverse Foldy-Wouthuysen transformation is applied to the canonical form of the Hamiltonian operator to provide a basis for the synthesis of invariant wave equations in the spirit of Foldy.[8] For spin 1/2 the transformed Hamiltonian operator reduces, of course, to the Dirac equation Hamiltonian. Although this is entirely expected, our work to this point could be viewed as a derivation of the Dirac equation from the fundamental principles formulated in Eqs. (A), (B), and (C). We are led to the Dirac equation by a logical series of steps beginning with solutions of Eqs. (A), (B), and (C).

For spin 1 the inverse Foldy-Wouthuysen transformed Hamiltonian gives a Schrödinger equation which is not local in coordinate space. Further manipulations are needed to get the local invariant Proca equations. As far as we know, the nonlocal equation has not previously appeared in the literature. From our point of view, it is more nearly the spin 1 analog of the Dirac equation than is the Schrödinger equation form of the Proca equations. We note that the Hamiltonian of the nonlocal equation is Hermitian while that of the Proca equations is pseudo-Hermitian. We compare the unitary generalized Foldy-Wouthuysen transformation with the pseudo-unitary transformation that takes the canonical form of the Hamiltonian directly to the Hamiltonian of the Proca equations. The latter does not have all the same properties as the former. In particular, its inverse takes the operators **x** that appear as independent variables in the Proca equations to operators that do not satisfy the last of Eqs. (B).

[7] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

[8] L. L. Foldy, Phys. Rev. **102**, 568 (1956).

II. PARTICLES WITH NO SPIN

Let us first consider a single particle of positive mass m and zero spin. From a mathematical point of view, we are interested in Hermitian operators H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$ which satisfy Eqs. (A) and generate the irreducible unitary representation of the inhomogeneous Lorentz group which is characterized by mass m, zero spin, and positive energy.[9] We are also interested in Hermitian operators $\mathbf{x}$ satisfying Eqs. (B) and (C). A canonical form for these operators is

$$H=(\mathbf{p}^2+\mathbf{m}^2)^{1/2}, \tag{2.1}$$

$$\mathbf{P}=\mathbf{p}, \tag{2.2}$$

$$\mathbf{J}=\mathbf{q}\times\mathbf{p}, \tag{2.3}$$

$$\mathbf{K}=\tfrac{1}{2}(H\mathbf{q}+\mathbf{q}H), \tag{2.4}$$

$$\mathbf{x}=\mathbf{q}, \tag{2.5}$$

where $\mathbf{q}$ and $\mathbf{p}$ are an irreducible set of Hermitian operators satisfying the commutation relations

$$\begin{aligned}&[q_j,q_k]=0=[p_j,p_k]\\&[q_j,p_k]=\delta_{jk},\end{aligned} \tag{2.6}$$

for j, $k=1,2,3$.

To within unitary equivalence, the operators (2.1)–(2.5) are the only solution of Eqs. (A) and (B) for a particle of mass m and zero spin. For, to within unitary equivalence, there is only one irreducible unitary representation of the inhomogeneous Lorentz group with positive mass m, zero spin, and positive energy[9]; by making a unitary transformation, we can always put H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$ equal to the operators (2.1)–(2.4). But, when H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$ have the canonical forms (2.1)–(2.4), the only solution for $\mathbf{x}$ of Eqs. (B) is the canonical coordinate (2.5).

These simple facts are included as a basis for normalizing our general procedures and for comparison with later results. We have seen that for a particle of mass m and zero spin Eqs. (A) and (B) determine the operators H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$, and $\mathbf{x}$ uniquely up to unitary equivalence. [Eq. (C) is then automatically satisfied in this case.] When H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$ have the canonical forms (2.1)–(2.4), $\mathbf{x}$ is equal to the canonical coordinate (2.5). All of the equations (B) are necessary for establishing this relation between the position variable $\mathbf{x}$ and the Lorentz group generators H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$.

The zero spin case is distinguished by the simplicity of the solution of Eqs. (A), (B), and (C). In this case all of Pryce's definitions (c), (d), (e) of position variables[6] coincide; they are all equal to the canonical position $\mathbf{x}=\mathbf{q}$ when H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$ have the canonical forms (2.1)–(2.4). The canonical position (2.5) is also the position operator found by Newton and Wigner.[5] If this is not obvious, it is because of a difference in normalization of wave functions. Our operators can be defined on momentum space wave functions with the inner product

$$(\phi,\psi)=\int\phi(\mathbf{p})^*\psi(\mathbf{p})d^3p. \tag{2.7}$$

Newton and Wigner work with wave functions normalized according to the invariant inner product

$$(\phi,\psi)=\int\phi(\mathbf{p})^*\psi(\mathbf{p})(\mathbf{p}^2+m^2)^{-1/2}d^3p. \tag{2.8}$$

The operators $\mathbf{q}$, which are Hermitian in the inner product (2.7), are not Hermitian in the inner product (2.8). The corresponding operators which are Hermitian in the inner product (2.8) are

$$\begin{aligned}\mathbf{x}&=(\mathbf{p}^2+m^2)^{1/4}\mathbf{q}(\mathbf{p}^2+m^2)^{-1/4}\\&=\mathbf{q}-i(1/2)(\mathbf{p}^2+m^2)^{-1}\mathbf{p}.\end{aligned} \tag{2.9}$$

These are the Newton-Wigner position operators. A similar operation yields canonical forms for H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$ that are appropriate for use with the inner product (2.8). The operators H, $\mathbf{P}$, $\mathbf{J}$ remain in the forms (2.1)–(2.3) but $\mathbf{K}$ becomes

$$\mathbf{K}=H\mathbf{q}. \tag{2.10}$$

One can check explicitly that Eqs. (2.1)–(2.3), (2.10), and (2.9) are solutions of Eqs. (A), (B), and (C) for H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$, and $\mathbf{x}$, respectively. For the zero spin case there is no problem in finding a position operator $\mathbf{x}$ satisfying the conditions (B) for relativistic covariance plus the conditions (C) of commuting components. Every approach seems to lead to the same answer. The complications occur with the introduction of spin.

III. PARTICLES WITH SPIN

We consider next a single particle of positive mass m and spin s, where s may have any one of the values 0, 1/2, 3/2, 2, $\cdots$. The relevant solution of Eqs. (A) consists of Hermitian operators H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$ which generate the irreducible unitary representation of the inhomogeneous Lorentz group which is characterized by mass m, spin s, and positive energy.[9] A canonical form for these operators is

$$H=(\mathbf{p}^2+m^2)^{1/2}, \tag{3.1}$$

$$\mathbf{P}=\mathbf{p}, \tag{3.2}$$

$$\mathbf{J}=\mathbf{q}\times\mathbf{p}+\mathbf{S}, \tag{3.3}$$

$$\mathbf{K}=\tfrac{1}{2}(H\mathbf{q}+\mathbf{q}H)+(H+m)^{-1}\mathbf{p}\times\mathbf{S}, \tag{3.4}$$

where $\mathbf{q}$, $\mathbf{p}$, and $\mathbf{S}$ are an irreducible set of Hermitian operators satisfying the commutation relations

$$\begin{aligned}&[q_i,q_k]=0=[p_j,p_k],\\&[q_j,S_k]=0=[p_j,S_k],\\&[q_j,p_k]=\delta_{jk},\\&[S_j,S_k]=\epsilon_{jkn}S_n,\end{aligned} \tag{3.5}$$

[9] E. P. Wigner, Ann. Math. **40**, 149 (1939); V. Bargmann, *ibid.* **48**, 568 (1947); V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. U. S. **34**, 211 (1948).

for j, k, $n=1,2,3$. These operators can be defined on a Hilbert space which is the direct product of the momentum space wave functions on which $\mathbf{q}$ and $\mathbf{p}$ are irreducible, and the $2s+1$ component spin vectors on which $\mathbf{S}$ generates an irreducible representation of the rotation group. This representation is characterized by the number

$$s(s+1)=\mathbf{S}^2=S_1^2+S_2^2+S_3^2. \qquad (3.6)$$

To within unitary equivalence, the operators (3.1)–(3.4) are the only solution of Eqs. (A) for a particle of mass m and spin s. For, to within unitary equivalence, there is only one irreducible unitary representation of the inhomogeneous Lorentz group with positive mass m, integral or half-integral spin s, and positive energy.[9] By making a unitary transformation, we can always put H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$ equal to the operators (3.1)–(3.4).

Now we want to look for operators $\mathbf{x}$ satisfying Eqs. (B) and (C). We do not try to find all possible solutions. We restrict ourselves to considering only operators that are linear in the spin variables $\mathbf{S}$. This means that we find all possible solutions only for the case $s=1/2$. We also find the solutions which are valid for all values of s and maintain their form as functions of $\mathbf{q}$, $\mathbf{p}$, and $\mathbf{S}$ independently of the value of s. In other words, for $s\geq 1$ we find those solutions which can also be made to be solutions for $s=1/2$ by simply reinterpreting $\mathbf{S}$ as spin 1/2 operators. [This is motivated by the fact that such a solution is the center of interest in the next section. We also note that the canonical solutions (3.1)–(3.4) of Eqs. (A) are of exactly this type.] We might overlook, for example, a solution for $s=1$ which contains terms quadratic in $\mathbf{S}$.

If H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$ have the canonical forms (3.1)–(3.4), the most general solution of Eqs. (B) for Hermitian operators $\mathbf{x}$ which are of at most linear order in the operators $\mathbf{S}$ is

$$\mathbf{x}=\mathbf{q}-aH^{-1}(H+m)^{-1}(\mathbf{p}\cdot\mathbf{S})\mathbf{p}+a\mathbf{S}-m^{-1}(H+m)^{-1}\mathbf{p}\times\mathbf{S}, \qquad (3.7)$$

where a is any real number. For $a=0$, this is the position operator (d) found by Pryce.[6] For any value of a, the operators (3.7) satisfy Eqs. (B) independently of the value of s. But for positive s there is no value of a for which the operators (3.7) satisfy Eqs. (C). For nonzero spin, and with the canonical forms (3.1)–(3.4) for H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$, there is no solution of Eqs. (B) and (C) for operators $\mathbf{x}$ which are of at most linear order in $\mathbf{S}$.

If we abandon the canonical forms (3.1)–(3.4) for H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$, we still cannot find a solution of Eqs. (B) and (C) which is valid for the case $s=1/2$. For suppose that H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$, and $\mathbf{x}$ satisfy Eqs. (B) for $s=1/2$. By making a unitary transformation, we can put H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$ equal to the canonical operators (3.4). We thus obtain a solution of Eqs. (B) with the canonical operators (3.1)–(3.4) and the unitary transform of $\mathbf{x}$. But the unitary transform of $\mathbf{x}$ must be of at most linear order in $\mathbf{S}$ since higher orders do not occur for $s=1/2$. Hence, the unitary transform of $\mathbf{x}$ must be one of the operators (3.7). It follows that the different components of $\mathbf{x}$ do not commute; we do not have a solution of Eqs. (C).

For a particle of positive mass there is no solution of Eqs. (A), (B), and (C) which is valid for all values of the spin; in particular, there is no solution for $s=1/2$. All of the equations (B) and (C) are necessary to produce this exclusion. With the canonical forms (3.1)–(3.4) for H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$, the operators (3.7) for $\mathbf{x}$ satisfy all except Eqs. (C) and the canonical position $\mathbf{x}=\mathbf{q}$ satisfies all except the last of Eqs. (B), the condition for Lorentz covariance. The canonical position $\mathbf{x}=\mathbf{q}$ is the same as the position variable (e) found by Pryce[6] and the position operator found by Newton and Wigner.[5] These authors have also concluded that it is not Lorentz covariant.

IV. PARTICLES AND ANTIPARTICLES

Let us now extend the scope of our investigation and consider a particle and antiparticle of positive mass m and spin s where s may have any one of the values 0, 1/2, 1, 3/2, 2 $\cdots$. The relevant solution of Eqs. (A) now consists of Hermitian operators H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$ which generate a direct sum of two irreducible unitary representations of the inhomogeneous Lorentz group, both having mass m and spin s, but one having positive and the other negative energy. A canonical form for these operators is

$$H=\rho_3(\mathbf{p}^2+m^2)^{1/2}=\rho_3 W, \qquad (4.1)$$

$$\mathbf{P}=\mathbf{p}, \qquad (4.2)$$

$$\mathbf{J}=\mathbf{q}\times\mathbf{p}+\mathbf{S}, \qquad (4.3)$$

$$\begin{aligned}\mathbf{K}&=\tfrac{1}{2}(H\mathbf{q}+\mathbf{q}H)+\rho_3(\rho_3 H+m)^{-1}\mathbf{p}\times\mathbf{S},\\ &=\tfrac{1}{2}\rho_3(W\mathbf{q}+\mathbf{q}W)+\rho_3(W+m)^{-1}\mathbf{p}\times\mathbf{S},\end{aligned} \qquad (4.4)$$

where $W=(\mathbf{p}^2+m^2)^{1/2}$ and where $\mathbf{q}$, $\mathbf{p}$, $\mathbf{S}$, and $\boldsymbol{\rho}$ are an irreducible set of Hermitian operators satisfying the commutation and anticommutation relations

$$\begin{aligned}
[q_j,q_k]&=0=[p_j,p_k],\\
[q_j,S_k]&=0=[p_j,S_k],\\
[q_j,\rho_k]&=0=[p_j,\rho_k],\\
[S_j,\rho_k]&=0,\\
[q_j,p_k]&=\delta_{jk},\\
[S_j,S_k]&=\epsilon_{jkn}S_n,\\
[\rho_j,\rho_k]&=2e_{jkn}\rho_n,\\
\rho_j\rho_k+\rho_k\rho_j&=2\delta_{jk},
\end{aligned} \qquad (4.5)$$

for j, k, $n=1,2,3$. These operators can be defined on a Hilbert space which is the direct product of the momentum space wave functions on which $\mathbf{q}$ and $\mathbf{p}$ are irreducible, the $2s+1$ component spin vectors on which $\mathbf{S}$ generates the irreducible representation of the rotation group characterized by the number $s(s+1)=\mathbf{S}^2$, and the two component vectors on which $\boldsymbol{\rho}$ are an irreducible set of Pauli matrices. We take ρ_3 to be diagonal:

$$\rho_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

This differs from the preceding section only in that we have doubled the number of components of our wave functions to allow for the negative energy or antiparticle states. The fact that, to within unitary equivalence, there is only one irreducible unitary representation of the inhomogeneous Lorentz group for mass m, spin s, and positive (negative) energy means that the canonical forms (4.1)–(4.4) for H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$ are unique, to within unitary equivalence, as functions of $\mathbf{q}$, $\mathbf{p}$, and $\mathbf{S}$. Their dependence on the operators $\boldsymbol{\rho}$ amounts to nothing more than the insertion of a minus sign factor in H and $\mathbf{K}$ for the antiparticle states.

Now, in contrast to the situation of the preceding section, we do have a solution for $\mathbf{x}$ of Eqs. (B) and (C) which is valid for the case $s=1/2$. It is the operator $\mathbf{x}$ which appears in the Dirac equation. More specifically, it is the Foldy-Wouthuysen representation[7] of the Dirac equation $\mathbf{x}$ which is in accord with the canonical forms (4.1)–(4.4) for H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$. In our notation (which is the same as that of Dirac's book[10]: $\boldsymbol{\alpha}=2\rho_1\mathbf{S}$, $\beta=\rho_3$), this is

$$\mathbf{x}=\mathbf{q}+\rho_2W^{-1}\mathbf{S}-\rho_2W^{-2}(W+m)^{-1}(\mathbf{p}\cdot\mathbf{S})\mathbf{p}+W^{-1}(W+m)^{-1}\mathbf{p}\times\mathbf{S}. \quad (4.6)$$

With the canonical forms (4.1)–(4.4) for H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$, the operators (4.6) are a solution for $\mathbf{x}$ of Eqs. (B) and (C), not only for the case $s=1/2$, but for all values of s!

Now let us see to what extent we have found a unique solution for $\mathbf{x}$. If we assume the canonical forms (4.1)–(4.4) for H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$, we find (after a certain amount of work) that the most general solution for $\mathbf{x}$ of Eqs. (B) and (C), for the case $s=1/2$, is

$$\begin{aligned}\mathbf{x}=\mathbf{q}&+\rho_1A\sin B\mathbf{p}+\rho_2A\cos B\mathbf{p}\\&-\rho_1(W^{-2}(W+m)^{-1}\sin B-2m^{-1}B'\cos B)(\mathbf{p}\cdot\mathbf{S})\mathbf{p}\\&-\rho_2(W^{-2}(W+m)^{-1}\cos B+2m^{-1}B'\sin B)(\mathbf{p}\cdot\mathbf{S})\mathbf{p}\\&+\rho_1W^{-1}\sin B\mathbf{S}+\rho_2W^{-1}\cos B\mathbf{S}\\&+W^{-1}(W+m)^{-1}\mathbf{p}\times\mathbf{S}, \quad (4.7)\end{aligned}$$

where A and B are arbitrary real functions of $\mathbf{p}^2$ and B' is the derivative of B with respect to $\mathbf{p}^2$.

If we allow operators H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$ that differ from the canonical forms (4.1)–(4.4) only as functions of $\mathbf{q}$, $\mathbf{p}$, and $\mathbf{S}$, we will find solutions for H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$, and $\mathbf{x}$ that are unitarily equivalent to the operators (4.1)–(4.4) and (4.7). As long as H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$ have the canonical dependence on $\boldsymbol{\rho}$, namely, a plus or minus sign factor in H and $\mathbf{K}$ for the particle and antiparticle states, we can always put them equal to the operators (4.1)–(4.4) by making a unitary transformation. In this sense we have found a solution that is unique to within unitary equivalence for the case $s=1/2$.

Further restrictions are necessary if the operators (4.7) are to be a solution for $\mathbf{x}$ of Eqs. (B) and (C) for values of s other than 1/2. In particular, in order for the operators (4.7) to satisfy Eqs. (C) for $s=1$ it is necessary and sufficient that $B'=0$. This implies that B is just a real number. Now ρ_1, ρ_2, and B occur in the operators (4.7) only in the combination

$$\rho_1\sin B+\rho_2\cos B.$$

If we choose an equivalent set of operators $\boldsymbol{\rho}$ in which this combination is called ρ_2 and in which ρ_3 is the same as before, we have

$$\mathbf{x}=\mathbf{q}+\rho_2A\mathbf{p}-\rho_2W^{-2}(W+m)^{-1}(\mathbf{p}\cdot\mathbf{S})\mathbf{p}+\rho_2W^{-1}\mathbf{S}+W^{-1}(W+m)^{-1}\mathbf{p}\times\mathbf{S}. \quad (4.8)$$

With the canonical forms (4.1)–(4.4) for H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$, the operators (4.8) for $\mathbf{x}$ satisfy Eqs. (B) and (C) for all values of s. This is the most general solution which is valid for all values of s and which maintains its form as functions of $\mathbf{q}$, $\mathbf{p}$, $\mathbf{S}$, and $\boldsymbol{\rho}$ independently of the value of s.

The Dirac position (4.6) is the particular case of the solution (4.8) for which $A=0$. Of the general solutions (4.8), the Dirac position (4.6) is the only one which reduces to the canonical coordinate $\mathbf{x}=\mathbf{q}$ when $\mathbf{S}=0$. In summary then, we have found that (assuming the canonical dependence of H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$ on $\boldsymbol{\rho}$) the Dirac position (4.6) is the unique solution [to within unitary transformations that change the form of H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$ as function of $\mathbf{q}$, $\mathbf{p}$, and $\mathbf{S}$ from that of the canonical operators (4.1)–(4.4)] of Eqs. (B) and (C) which is valid and has the same form for all values of s and reduces to the canonical coordinate $\mathbf{x}=\mathbf{q}$ when $s=0$.

We note that this position operator retains also the strange properties of the spin-1/2 Dirac position operator. In particular, it gives a velocity which has no sensible physical interpretation. It is impossible to satisfy both Eqs. (B) for relativistic covariance and all of the other properties that one might expect of a position operator.

We were led to the position operator (4.6) by recognizing that for $s=1/2$ we had a solution of Eqs. (B) and (C) by the operator $\mathbf{x}$ which appears in the Dirac equation. We can regain the Dirac equation by making an inverse Foldy-Wouthuysen transformation[7] which will take the canonical Hamiltonian (4.1) to the Dirac Hamiltonian. We prefer to state this in a way that illuminates the role of the covariant position operator. The inverse Foldy-Wouthuysen transformation is just the unitary transformation which takes the Dirac position operator (4.6) to the canonical coordinate $\mathbf{x}=\mathbf{q}$ while leaving the canonical forms (4.2) and (4.3) for $\mathbf{P}$ and $\mathbf{J}$ unchanged. If

$$V=-\rho_2p^{-1}(\mathbf{p}\cdot\mathbf{S})\tan^{-1}(p/m) \quad (4.9)$$

(where $p^2=\mathbf{p}^2$) and if $\mathbf{x}$ is the Dirac position (4.6), then[7]

$$e^{iV}\mathbf{x}e^{-iV}=\mathbf{q} \tag{4.10}$$

$$e^{iV}\mathbf{p}e^{-iV}=\mathbf{p} \tag{4.11}$$

$$e^{iV}(\mathbf{q}\times\mathbf{p}+\mathbf{S})e^{-iV}=\mathbf{q}\times\mathbf{p}+\mathbf{S}. \tag{4.12}$$

Furthermore, the unitary transformation e^{iV}, with V the operator (4.9), is essentially the unique unitary transformation with the properties (4.10)–(4.12). For any additional unitary transformation would have to leave $\mathbf{q}$, $\mathbf{p}$, and $\mathbf{S}$ unchanged and so could only be a transformation of the operators $\boldsymbol{\rho}$. These statements are all valid for all values of s, not just for the case $s=1/2$. The inverse Foldy-Wouthuysen transformation, just like the Dirac position operator (4.6), maintains its essential properties under a generalization from spin 1/2 to any integral or half-integral spin.

V. THE SPIN $\frac{1}{2}$ AND SPIN 1 EQUATIONS

Under the generalized inverse Foldy-Wouthuysen transformation with the operator (4.9), which is applicable for all values of s, the canonical Hamiltonian (4.1) becomes

$$e^{iV}\rho_3We^{-iV}=W\{\rho_3\cos[2p^{-1}(\mathbf{p}\cdot\mathbf{S})\tan^{-1}(p/m)] +\rho_1\sin[2p^{-1}(\mathbf{p}\cdot\mathbf{S})\tan^{-1}(p/m)]\}, \tag{5.1}$$

where again $p^2=\mathbf{p}^2$ and $W=(\mathbf{p}^2+m^2)^{1/2}$.

Spin $\frac{1}{2}$

For the case $s=1/2$ the operator (5.1) reduces to

$$\rho_3m+\rho_1 2(\mathbf{p}\cdot\mathbf{s})=\beta m+\boldsymbol{\alpha}\cdot\mathbf{p} \tag{5.2}$$

which is the Hamiltonian of the Dirac equation. Here we have simply regained the familiar Foldy-Wouthuysen spin 1/2 transformation[7] which has been the basic motivation for our more general statements. But in a certain sense we may regard what we have done as a derivation of the Dirac equation from fundamental postulates of relativistic invariance. We began with the canonical forms for the generators H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$ of an irreducible unitary representation of the inhomogeneous Lorentz group with mass m and spin s and looked for position operators $\mathbf{x}$ satisfying the conditions (B) for relativistic covariance and the conditions (C) of commuting components. Being unable to find a solution for $\mathbf{x}$ on the states of a single particle, we enlarged the scope of our study to include the antiparticle states. We then found the generalized Dirac position (4.6) as a rather unique solution for $\mathbf{x}$. The generalized inverse Foldy-Wouthuysen transformation appeared as the unitary transformation which changes the Dirac position (4.6) into the canonical coordinate $\mathbf{x}=\mathbf{q}$ without changing the canonical forms (4.2) and (4.3) for $\mathbf{P}$ and $\mathbf{J}$. Under this transformation the canonical Hamiltonian (4.1) goes into the Dirac equation Hamiltonian (5.2) for the case $s=1/2$. It makes sense that in looking for a manifestly invariant local wave equation we should transform to a representation in which the covariant position operator is the canonical coordinate. But, as we presently see, this does not guarantee the desired local equation. Nevertheless, had we known nothing about the Dirac equation, we could have found it by the method just outlined.

Spin 1

For the case $s=1$, the Hamiltonian operator (5.1) reduces to

$$\rho_3W-\rho_3 2W^{-1}(\mathbf{p}\cdot\mathbf{S})^2+\rho_1 2mW^{-1}(\mathbf{p}\cdot\mathbf{S}). \tag{5.3}$$

This gives a Schrödinger wave equation

$$i(\partial/\partial t)\psi=H\psi, \tag{5.4}$$

with H the operator (5.3), $\boldsymbol{\rho}$ and $\mathbf{S}$ six by six matrices, and ψ a six-component wave function. In contrast to the Dirac equation, this equation is not local in coordinate space. This is not surprising because we expect that for integral spin a local invariant equation must involve an indefinite metric. We note that the Hamiltonian (5.3) is Hermitian.

If we make the identification

$$\psi=\begin{pmatrix}\mathbf{F}\\ \mathbf{G}\end{pmatrix}$$

$$\begin{aligned}\mathbf{F}&=W^{-1}(2W)^{-1/2}\{m(\mathbf{E}+iW\mathbf{A})-\nabla\times(\mathbf{E}-iW\mathbf{A}) -m^{-1}\nabla(\nabla\cdot\mathbf{E})\}\\ \mathbf{G}&=W^{-1}(2W)^{-1/2}\{m(\mathbf{E}-iW\mathbf{A})+\nabla\times(\mathbf{E}+iW\mathbf{A}) -m^{-1}\nabla(\nabla\cdot\mathbf{E})\}\end{aligned} \tag{5.5}$$

we find that the Schrödinger equation (5.4) with the Hamiltonian (5.3) is equivalent to the Proca equations[8,11]

$$\begin{aligned}(\partial/\partial t)\mathbf{A}&=-\mathbf{E}-\nabla\phi,\\ (\partial/\partial t)\mathbf{E}&=m^2\mathbf{A}+\nabla\times\mathbf{B},\\ \phi&=-m^{-2}\nabla\cdot\mathbf{E},\\ \mathbf{B}&=\nabla\times\mathbf{A},\end{aligned} \tag{5.6}$$

which are local invariant wave equations for spin 1. [Here we have used the standard Pauli matrices for $\boldsymbol{\rho}$ and the standard spin 1 matrices $(S_j)_{kn}=i\epsilon_{jkn}$.]

The generalized inverse Foldy-Wouthuysen transformation by itself does not take us to local invariant equations for spin 1; it gives us a Schrödinger equation (5.4) with the Hamiltonian (5.3). The further manipulations (5.5) are needed to get the local invariant equations (5.6). This is not surprising. The same sort of thing happens for zero spin. For zero spin the generalized inverse Foldy-Wouthuysen transformation is

[10] P. A. M. Dirac, *Principles of Quantum Mechanics* (Oxford University Press, Oxford, England, 1958).

[11] G. Wentzel, *Quantum Theory of Fields* (Interscience Publishers, Inc., New York, 1949), Chap. III.

just the identity transformation; the generalized covariant Dirac position operator is just the canonical coordinate $\mathbf{x}=\mathbf{q}$. So the generalized inverse Foldy-Wouthuysen transformation leaves the Hamiltonian in the canonical form (4.1) for zero spin. But considerably more manipulation is needed to get the Klein-Gordon equation.[8]

The equations (5.6) can be put directly into the Schrödinger equation form (5.4) with a six-component wave function whose components are $\mathbf{A}$ and $m^{-1}\mathbf{E}$.[8,11,21] The Hamiltonian for this equation is not Hermitian but is pseudo-Hermitian in the appropriate indefinite metric.[13] From our point of view, the Schrödinger equation with the Hamiltonian (5.3), which is obtained by the generalized inverse Foldy-Wouthuysen transformation from the canonical Hamiltonian (4.1), is more nearly the spin 1 analog of the Dirac equation. Whether it will be more useful remains to be seen.

Various authors have developed the transformation which connects the canonical Hamiltonian (4.1) directly to the Schrödinger equation form of Eqs. (5.6) with the wave function whose components are $\mathbf{A}$ and $m^{-}\mathbf{E}$.[8,13] This is not a unitary transformation but is pseudo-unitary in the appropriate indefinite metric.[13] From our point of view, it appears as the combination of the generalized inverse Foldy-Wouthuysen transformation and the manipulations (5.5). This transformation does not have all of the properties of the Foldy-Wouthuysen transformation and cannot be put to all of the same uses. For example, if we use it to transform the position operators $\mathbf{x}$ which appear as the independent variables in the equations (5.6) to the representation in which the Hamiltonian has the canonical form (4.1), we get an operator[14] which does not satisfy the last of Eqs. (B), the condition for Lorentz covariance.

Finally, we want to point out that we have exposed several simple features of the spin 1/2 situation which are not shared by spin 0 or 1. In particular, the inverse Foldy-Wouthuysen transformation for $s=1/2$ gives us the local invariant Dirac equation. The analogous equations for $s=0$ and 1 are not local.

ACKNOWLEDGMENT

We are grateful to Professor E. C. G. Sudarshan for helpful discussions of this work.

[12] E. M. Corson *Tensors, Spinors, and Relativistic Wave Equations* (Hafner Publishing Company, New York, 1953), especially Secs. 26(b) and 39(d)(i).

[13] K. M. Case, Phys. Rev. **95**, 1323 (1954).

[14] K. M. Case, Ref. 13, Eq. (38).

PHYSICAL REVIEW VOLUME 132, NUMBER 4 15 NOVEMBER 1963

Improvement of the Born Series at Low Energy*

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(Received 8 July 1963)

The Born series for a quantum-mechanical Green function is studied. A prescription is given for making "best" use of the information contained in the first few terms of that series, and, in particular, for calculating bound states or resonances from them. This prescription is based on heuristic convergence arguments whose formal steps are somewhat reminiscent of renormalization group methods. The present considerations may be applied to potential scattering as well as to quantum field theory. They are expected to be valid for low-energy phenomena and finite-range forces. The prescription is tested, using only the first two Born terms, in the case of a nonrelativistic particle moving in a Yukawa potential: For well depths producing a single shallow bound state, the usual effective-range results are closely reproduced, and, in some ways, improved upon.

1. INTRODUCTION

The problem of replacing the quantum-mechanical Born (or perturbation) series by a more convergent expansion is already a well-studied one. Its importance arises from the fact that many cases of great physical interest cannot be treated by means of that series. Bound-state and resonance problems fall in this category whenever the unperturbed problem yields only a continuum of noninteracting states. This is precisely the situation one must face in relativistic field theory. The current experimental results with strongly interacting particles only emphasize the need for improved calculational procedures in this area.

Among the many existing approaches[1–3] to the ques-

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[1] See, for example, P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Part II, Chap. 9.

[2] For recent approaches see M. Rotenberg, Ann. Phys. (N. Y.) **21**, 579 (1963), and its bibliography. Rotenberg's method, like the present one, is based on a regrouping of the Born terms, designed to accelerate convergence. His "regrouped" Eq. (59), in particular, should be compared with our formulation, in which a true regrouping only occurs as an intermediate step.

[3] A method of circumventing the convergence difficulty of